Computing the Real Stability Radius for Frobenius-norm Bounded Perturbations

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Motivation: Robust Stability

- Linear system $x'(t) = Ax(t)$ is stable if $x(t) \to 0$ for all starting points. Requires eigs of A have negative real part.
- Measure sensitivity (robustness) of stability to perturbations as:
  - Real-life control systems are subject to noise and uncertainties in modeling.
  - Input noise should not be amplified much in applications.
  - Designing control systems by optimizing robustness measures.

Controller for Boeing 767

International Space Station

Pseudospectra: Complex Perturbations (Reminder)

Subject of previous talk: Complex 2-norm bounded perturbations Δ.

$$\begin{aligned}\sigma_\varepsilon(A) &= \bigcup\{\sigma(A+\Delta) : ||\Delta|| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{C} : \underbrace{||(\lambda I - A)^{-1}||_2}_{:=\eta_A(\lambda) \text{ norm of the resolvent}} \geq \frac{1}{\varepsilon}\}\end{aligned}$$

- $\varepsilon = 10^{-1.6}$
- $\varepsilon = 10^{-1.2}$
- $\varepsilon_* = 10^{-0.8}$
  $\alpha_{\varepsilon_*}(A) = 0$

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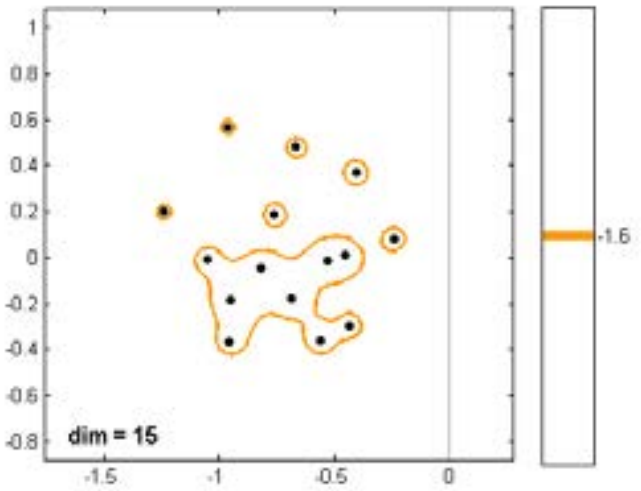

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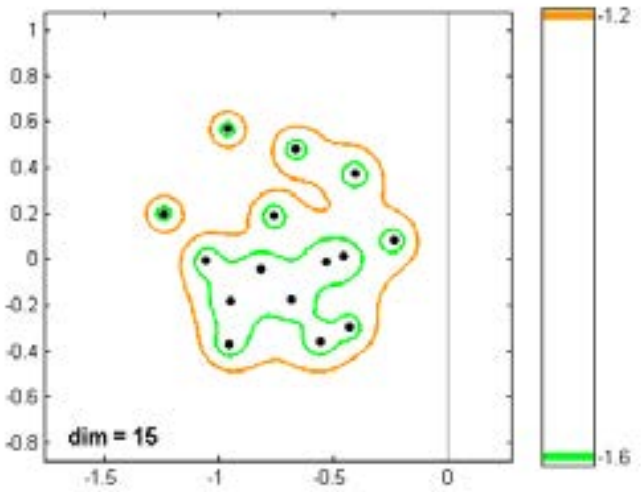

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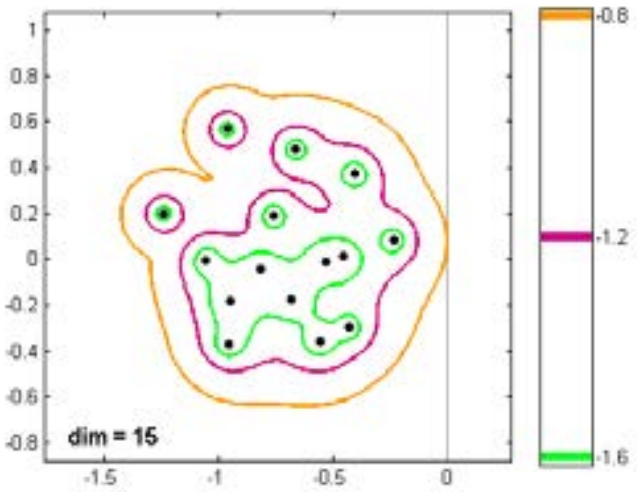

Real/Complex Spectral Value Sets

- Spectral value sets are generalizations of pseudospectra:

$$\sigma_\varepsilon^{\mathbb{K},\|\cdot\|}(A,B,C,D) = \bigcup \left\{ \sigma(M(\Delta)) : \Delta \in \mathbb{K}^{p\times m}, \|\Delta\| \le \varepsilon \right\}.$$

$$M(\Delta) = A + B\Delta(I - D\Delta)^{-1}C \quad \text{for } \Delta \in \mathbb{K}^{p\times m}$$

- Different perturbation classes e.g. and norms to measure it:
  $\|\cdot\| = \|\cdot\|_2, \|\cdot\|_{\mathrm{F}}, \ldots, \quad \mathbb{K} = \mathbb{C} \text{ or } \mathbb{R}.$

Key Lemma. Let $\varepsilon \in \mathbb{R}^+$ such that $\varepsilon\|D\| < 1$. Then

$$\sigma_\varepsilon^{\mathbb{R},\|\cdot\|}(A,B,C,D) \equiv \bigcup \left\{ \sigma(M(\Delta)) : \Delta \in \mathbb{R}^{p\times m}, \|\Delta\| \le \varepsilon, \operatorname{rank}(\Delta) \le 2 \right\}$$
$$\equiv \bigcup \left\{ \lambda \in \mathbb{C} : \mu^{\mathbb{R},\|\cdot\|}(G(\lambda)) \ge \varepsilon^{-1} \right\},$$

where $G(\lambda) = C(\lambda I - A)^{-1}B + D$ is the transfer matrix function, $\mu^{\mathbb{R},\|\cdot\|}$ is the real structured singular value, [Doyle 1982].

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Robust Stability Measures of Interest

- Many control systems are subject to noise that is real-valued, motivates computing robust stability measures under real-valued perturbations.
- For $\varepsilon \geq 0$, $\varepsilon\|D\| < 1$, the real structured spectral value set abscissa is

$$\alpha_{\varepsilon}^{\mathbb{R},\|\cdot\|}(A,B,C,D) := \max\left\{\operatorname{Re}(\lambda) : \lambda \in \sigma_{\varepsilon}^{\mathbb{R},\|\cdot\|}(A,B,C,D)\right\}$$

- The real stability radius for the system described by (A,B,C,D) is

$$\varepsilon_{\star}^{\mathbb{R},\|\cdot\|} := \sup\left\{\varepsilon : \varepsilon\|D\| < 1 \text{ and } \sigma_{\varepsilon}^{\mathbb{R},\|\cdot\|}(A,B,C,D) \subset \mathbb{C}_{-}\right\}.$$

  where $\mathbb{C}_{-}$ = open left half-plane.
- By definition, the stability radius $\varepsilon_{\star}^{\mathbb{R},\|\cdot\|}$ is the zero of the monotonic function $g^{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ defined by

$$g^{\mathbb{R}}(\varepsilon) = \alpha_{\varepsilon}^{\mathbb{R},\|\cdot\|}(A,B,C,D)$$

Vast Literature: A Partial List

The Pseudospectra Case ($B = C = I$, $D = 0$)

- Mengi-Overton 2004: stability radius in discrete-time for $\mathbb{K} = \mathbb{C}$
- Guglielmi-Overton 2011: a discrete iteration for $\mathbb{K} = \mathbb{C}$
- Kressner-Vandereycken 2012: improved method for $\mathbb{K} = \mathbb{C}$
- Guglielmi-Lubich 2013: an ODE approach for $\mathbb{K} = \mathbb{R}$
- Guglielmi-Manetta 2014: a discrete iteration for $\mathbb{K} = \mathbb{R}$
- Freitag-Spence 2015: implicit determinant approach for $\mathbb{K} = \mathbb{R}$
- Kostic-Miedlar-Stolwick 15: general stability regions and $\mathbb{K} = \mathbb{C}$

Extensions to Spectral Value Sets (general A, B, C, D)

- Gürbüzbalaban-Guglielmi-Overton 2013: a discrete iteration for complex spectral value sets and a Newton-bisection method for the complex stability radius (and H_∞ norm), $\mathbb{K} = \mathbb{C}$
- Benner-Voigt 2014: a closely related method, $\mathbb{K} = \mathbb{C}$
- Mitchell-Overton 2014: a hybrid expansion contraction (HEC) method for the complex stability radius (and H_∞ norm), $\mathbb{K} = \mathbb{C}$

Our Contribution

- We focus on computing the real stability radius and real spectral value set abscissa.
- In 1995, an algorithm was given to extend to compute $\varepsilon_\star^{\mathbb{R},\|\cdot\|_2}$ to guaranteed precision [Qiu et al. 95, Sreedhar-Van Dooren-Tits 96].
- More work than the BBBS alg.; no good for large scale, does not work for F-norm.
- Our contribution: Developed Algorithm RSVSA to approximate $\varepsilon_\star^{\mathbb{R},\|\cdot\|_{\mathrm{F}}}$, extending Algorithm SVSA (described in the previous talk).
- Key point: Iterate with rank-two perturbations instead of rank-one. Inspired by [Guglielmi-Lubich 2013, Guglielmi-Manetta 2014].

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How to Make Perturbations Evolve in Time?

Lemma

Given a smooth parametrization $\Delta(t)$ with $||\Delta(t)||_{\mathrm{F}} = \varepsilon$ for all $t \geq 0$, suppose $\lambda(t)$ is the unique rightmost eigenvalue of $M(\Delta(t))$ and assume it is simple for all $t \geq 0$. Then $\lambda(t)$ is differentiable with

$$\begin{aligned}\operatorname{Re}\dot{\lambda}(t) &= \frac{1}{y(t)^* x(t)} \operatorname{Re}(u(t)^* \dot{\Delta}(t) v(t)) \\ &= \frac{1}{y(t)^* x(t)} \left\langle \dot{\Delta}(t), \operatorname{Re}(u(t) v(t)^*) \right\rangle\end{aligned}$$

where $x(t)$ and $y(t)$ are respectively right and left RP-compatible eigenvectors associated with $\lambda(t)$, i.e., with $\|x(t)\| = \|y(t)\| = 1$ and $y(t)^* x(t)$ real and positive, and

$$u(t) = (I - \Delta(t)D)^{-\mathsf{T}} B^{\mathsf{T}} y(t), \quad v(t) = (I - D\Delta(t))^{-1} C x(t).$$

How to Make Perturbations Evolve in Time?

- How to choose the evolution of $\Delta(t)$?
  - Move to the right as quick as possible, i.e. select $\dot{E}(t)$ to $\max \mathrm{Re}(\dot{\lambda}(t))$.
  - Need the magnitude right. Set $\Delta(t) = \varepsilon E(t)$ and keep $\|E(t)\|_{\mathrm{F}} = 1$.
- Explicit solution to these two criteria leads to the ODE:

$$\dot{E}(t) = \mathrm{Re}\left(u(t)v(t)^*\right) - \left\langle E(t), \mathrm{Re}\left(u(t)v(t)^*\right)\right\rangle E(t),$$

with $u(t)$ and $v(t)$ defined by

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The Solution of the ODE

Theorem

Suppose for simplicity that for all $t \geq 0$, the rightmost eigenvalue $\lambda(t)$ of $M(E(t))$ is unique and simple. Then, $\mathrm{Re}\ \dot{\lambda}(t) \geq 0$ for all t and, as $t \to \infty$:

1. *$\lambda(t)$ converges to a limit $\tilde{\lambda}$.*
2. *$\mathrm{Re}\,(u(t)v(t)^*)$ converges to a limit $\mathrm{Re}\,(\tilde{u}\tilde{v}^*)$.*
3. *$E(t)$ converges to an "optimal" perturbation, i.e.*

$$E(t) \to \tilde{E} := \frac{\mathrm{Re}\,(\tilde{u}\tilde{v}^*)}{\|\,\mathrm{Re}\,(\tilde{u}\tilde{v}^*)\,\|_{\mathrm{F}}}$$

with $\tilde{\lambda}$ a rightmost eigenvalue of $M(\varepsilon\tilde{E})$ if $\mathrm{Re}\,(\tilde{u}\tilde{v}^*) \neq 0$.

Roughly speaking, if $\tilde{\lambda}$ is the unique simple rightmost eigenvalue, "There is no locally differentiable path passing from $\tilde{E}$ that could let you move to the right further with positive speed."

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The Rank of the Desired Perturbation $\tilde{E}$

By the previous theorem, the perturbation $\tilde{E} \propto \mathrm{Re}(\tilde{u}\tilde{v}^*)$ we desire satisfies

$$\tilde{u} = \left(I - \varepsilon \tilde{E} D\right)^{-\mathsf{T}} B^{\mathsf{T}} \tilde{y}, \ \tilde{v} = \left(I - \varepsilon D \tilde{E}\right)^{-1} C \tilde{x}$$

where $\tilde{y}$ and $\tilde{x}$ eigenvectors of $M(\varepsilon \tilde{E})$ corresponding to $\tilde{\lambda}$.

1. If $\tilde{\lambda} \in \mathbb{R}$, then $\operatorname{rank} \mathrm{Re}(\tilde{u}\tilde{v}^*) = 1$ as we can choose $\tilde{y}, \tilde{x}$ to be real.
2. If $\tilde{\lambda} \notin \mathbb{R}$, set

$$\tilde{X} = (\mathrm{Re}\ \tilde{x}, \mathrm{Im}\ \tilde{x}) \in \mathbb{R}^{n \times 2}, \tilde{Y} = (\mathrm{Re}\ \tilde{y}, \mathrm{Im}\ \tilde{y}) \in \mathbb{R}^{n \times 2}$$

so $\mathrm{Re}(\tilde{y}\tilde{x}^*) = \tilde{Y}\tilde{X}^{\mathsf{T}}$. Then

$$\mathrm{Re}(\tilde{u}\tilde{v}^*) = \underbrace{\left(I - \varepsilon \tilde{E} D\right)^{-\mathsf{T}} B^{\mathsf{T}} \tilde{Y}}_{=:\tilde{U}} \underbrace{\tilde{X}^{\mathsf{T}} C^{\mathsf{T}} \left(I - \varepsilon D \tilde{E}\right)^{-\mathsf{T}}}_{=:\tilde{V}^{\mathsf{T}}}$$

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Solving the ODE to Approximate $\alpha_\varepsilon^{\mathbb{R},||\cdot||_{\mathrm{F}}}$

- We could integrate the ODE numerically with automatic stepsize control, but the solution $E(t)$ and its discretization will not preserve the low rank structure even if the rank is two initially and in the limit.
- Following [Guglielmi-Manetta 2014], consider the following implicit-explicit Euler discretization of the ODE with a step h_k:

$$E_k = E_{k-1} + h_k\Big(\mathrm{Re}(u_k v_k^*) - \big\langle E_k, \mathrm{Re}(u_k v_k^*)\big\rangle E_{k-1}\Big)$$

where u_k and v_k are given by

$$u_k = \big(I - \varepsilon E_{k-1} D\big)^{-\mathsf{T}} B^{\mathsf{T}} y_{k-1}, \quad v_k = \big(I - \varepsilon D E_{k-1}\big)^{-1} C x_{k-1}$$

and y_{k-1} and x_{k-1} are RP-compatible eigenvectors associated with the rightmost eigenvalue λ_{k-1} of $M(\varepsilon E_{k-1})$.

- The method is consistent and converges with order 1 with respect to h_k.

Solution of the Difference Equation

- Let u_0, v_0 be given vectors, let $E_0 = \mathrm{Re}\,(u_0 v_0^*)\,/\|\,\mathrm{Re}\,(u_0 v_0^*)\,\|_{\mathrm{F}}$ and $h_k = 1/\|\,\mathrm{Re}\,(u_k v_k^*)\,\|_{\mathrm{F}}$. Then the difference equation has the solution[1]

$$E_k = \frac{\mathrm{Re}\,(u_k v_k^*)}{\|\,\mathrm{Re}\,(u_k v_k^*)\,\|_{\mathrm{F}}}.$$

- So, $\mathrm{rank}(E_k) \leq 2$ and the iteration can be done very efficiently. The main cost is computing λ_k, x_k and y_k (by 2 calls to `eigs`).
- There is no guarantee that $\mathrm{Re}(\lambda_k) \geq \mathrm{Re}(\lambda_{k-1})$, but this is generally expected to hold since the ODE discretized by the iteration has this monotonicity property. In any case, we can modify the iteration by introducing a "line search" to ensure that monotonicity holds.

[1] as long as $\mathrm{Re}\,(u_k v_k^*) \neq 0$ and the rightmost eigenvalue of $M(\varepsilon E_k)$ is unique and simple for all $k \geq 0$.

Iteration RSVSA (Real Spectral Value Set Abscissa)

Input: ε, A, B, C, D, E_0

For $k = 1, 2, \ldots,$ let λ_{k-1} be the rightmost eigenvalue of $M(\varepsilon E_{k-1})$ with corresponding RP-compatible right and left eigenvectors x_{k-1} and y_{k-1}, and set[2]

$$
\begin{aligned}
X_{k-1} &= (\mathrm{Re}\ x_{k-1}, \mathrm{Im}\ x_{k-1}) \in \mathbb{R}^{n\times 2} \\
Y_{k-1} &= (\mathrm{Re}\ y_{k-1}, \mathrm{Im}\ y_{k-1}) \in \mathbb{R}^{n\times 2} \\
U_k &= (I - \varepsilon E_{k-1} D)^{\mathsf{T}} B^{\mathsf{T}} Y_{k-1} \in \mathbb{R}^{p\times 2} \\
V_k &= (I - \varepsilon D E_{k-1})\, C X_{k-1} \in \mathbb{R}^{m\times 2} \\
\beta_k &= 1/\|U_k V_k^{\mathsf{T}}\|_{\mathrm{F}} \in \mathbb{R} \\
E_k &= \beta_k U_k V_k^{\mathsf{T}} \in \mathbb{R}^{p\times m}
\end{aligned}
$$

[2] well-defined if the rightmost eigenvalue of $M(\varepsilon E_{k-1})$ is unique and simple and $U_k V_k^{\mathsf{T}}$ is nonzero.

Iteration RSVSA (Real Spectral Value Set Abscissa)

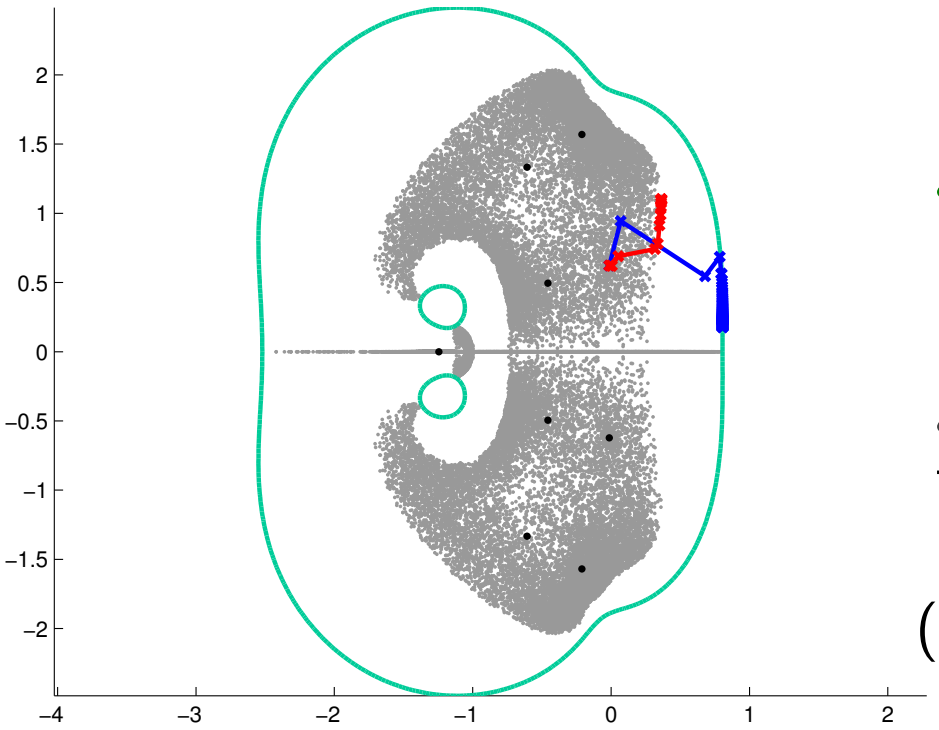
Input: ε, A, B, C, D, E_0

For $k = 1, 2, \ldots,$ let λ_{k-1} be the rightmost eigenvalue of $M(\varepsilon E_{k-1})$ with corresponding RP-compatible right and left eigenvectors x_{k-1} and y_{k-1}, and set[2]

$$
\begin{aligned}
X_{k-1} &= (\mathrm{Re}\ x_{k-1}, \mathrm{Im}\ x_{k-1}) \in \mathbb{R}^{n\times 2} \\
Y_{k-1} &= (\mathrm{Re}\ y_{k-1}, \mathrm{Im}\ y_{k-1}) \in \mathbb{R}^{n\times 2} \\
U_k &= (I - \varepsilon E_{k-1} D)^{\mathsf{T}} B^{\mathsf{T}} Y_{k-1} \in \mathbb{R}^{p\times 2} \\
V_k &= (I - \varepsilon D E_{k-1})\, C X_{k-1} \in \mathbb{R}^{m\times 2} \\
\beta_k &= 1/\| U_k V_k^{\mathsf{T}} \|_{\mathrm{F}} \in \mathbb{R} \\
E_k &= \beta_k U_k V_k^{\mathsf{T}} \in \mathbb{R}^{p\times m}
\end{aligned}
$$

[2] well-defined if the rightmost eigenvalue of $M(\varepsilon E_{k-1})$ is unique and simple and $U_k V_k^{\mathsf{T}}$ is nonzero.

Iterations of Algorithm RSVSA: Example 1



black: $\sigma(A)$

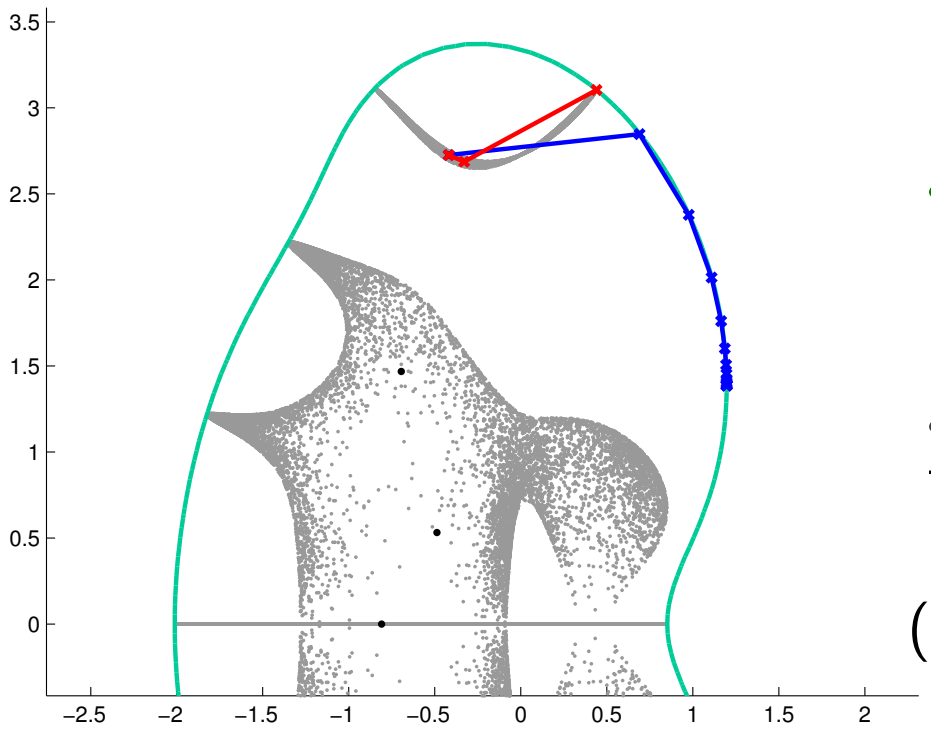
green: boundary of $\sigma_\varepsilon(A, B, C, D)$

blue: iterates of Algorithm SVSA (complex perturbations)

gray: $\sigma_\varepsilon^{\mathbb{R},\|\cdot\|_{\mathrm{F}}}(A, B, C, D)$, sampled from 6000 random perturbations

red: iterates of Algorithm RSVSA (real perturbations, Frob. norm)

Iterations of Algorithm RSVSA: Example 2



black: $\sigma(A)$

green: boundary of $\sigma_\varepsilon(A, B, C, D)$

blue: iterates of Algorithm SVSA (complex perturbations)

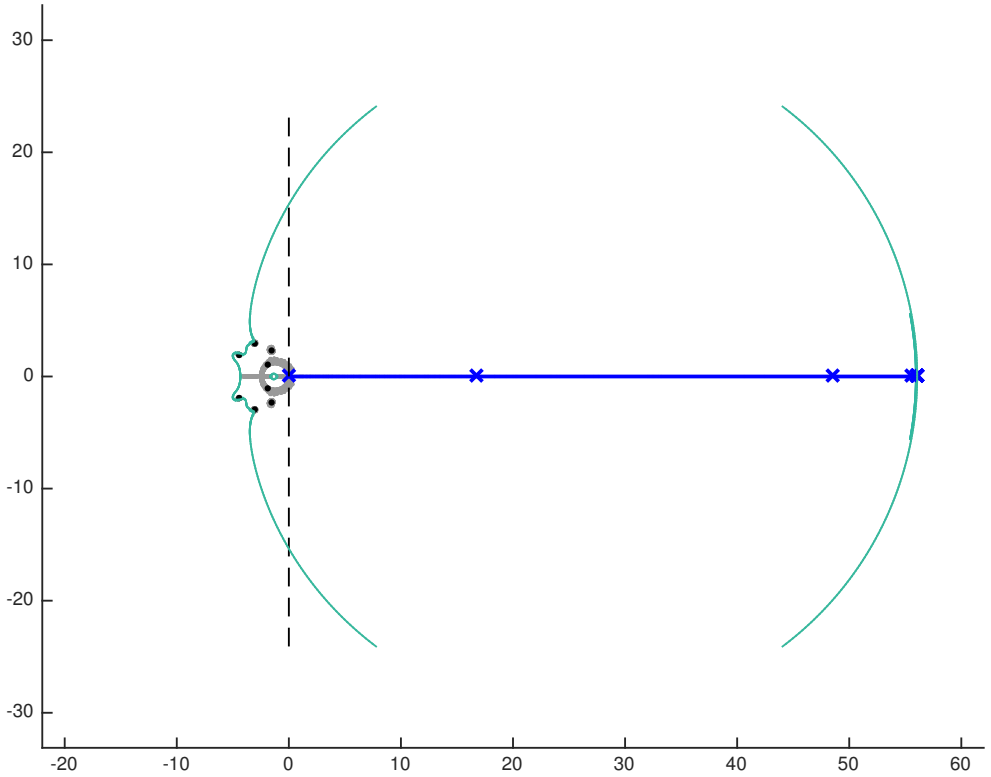
gray: $\sigma_\varepsilon^{\mathbb{R},\|\cdot\|_{\mathrm{F}}}(A, B, C, D)$, sampled from 6000 random perturbations

red: iterates of Algorithm RSVSA (real perturbations, Frob. norm)

Computing the real stability radius with HEC

- By definition, the real stability radius ε^* is equal to the ε level which results in $\alpha_\varepsilon^{\mathbb{R},\|\cdot\|_F}(A,B,C,D) = 0$.
- Now that we have an efficient alg. to compute $\alpha_\varepsilon^{\mathbb{R},\|\cdot\|_F}(A,B,C,D)$ for a given ε, we can use the Hybrid Expansion-Contraction (HEC) approach from [Mitchell-Overton 2014] to compute ε^* which extends with minor modifications.
- Recap from previous talk: HEC improves the bisection-Newton approach with some desirable properties:
  - Adaptively damped/accelerated Newton scheme, provably quadratically convergent that is visible in the experiments.
  - Often/usually finds the true value of the H-infinity norm.
  - When it does not, usually finds good approximations.
  - Also very fast in practice.

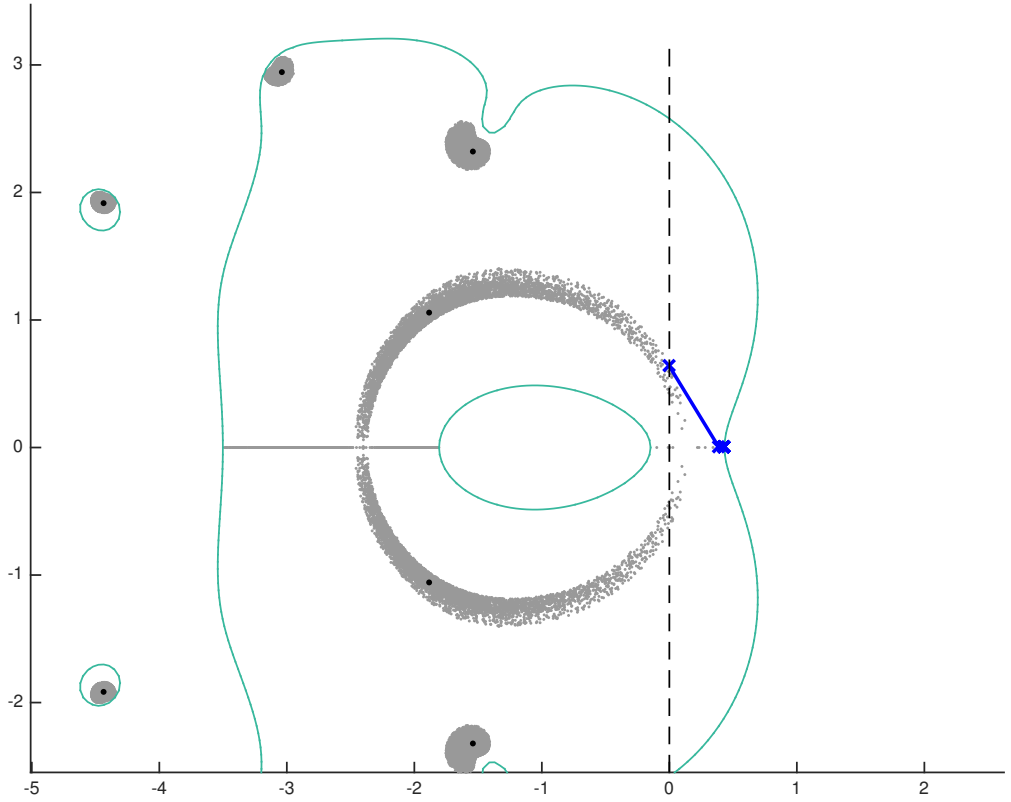
Example HEC Iterates - Step 1



Step 0: Generate a point in the right half-plane and contract it back to the imaginary axis, gives an upper bound $\varepsilon_1 \approx 0.71$ of the real stability radius $\varepsilon_\star$.

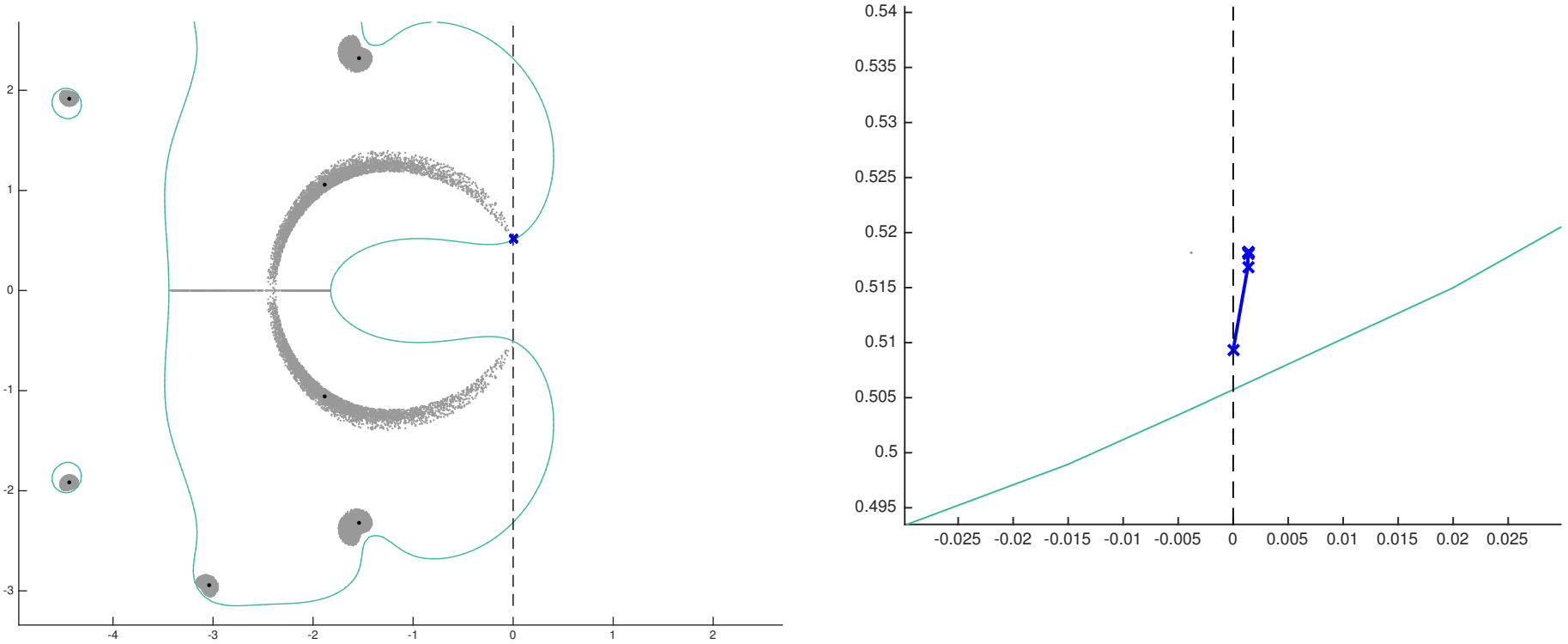
Step 1: Expand to compute abscissa, $\alpha_{\varepsilon_1} \approx 56.06$.

Example HEC Iterates - Step 2



Step 2: Contracting previous iterate to imaginary axis, gives a new upper bound $\varepsilon_2 \approx 0.42$, iterates shown for computing $\alpha_{\varepsilon_2} \approx 0.43$.

Example HEC Iterates - Step 3



Step 3: Left: Contracting previous iterate gives a new upper bound $\varepsilon_3 \approx 0.40$, iterates shown for computing $\alpha_{\varepsilon_3} \approx 0.0013$. Right: Zoom in.

Summary and future work

- Presented a new algorithm for computing the real spectral value set abscissa and real stability radius under Frobenius-norm bounded perturbations.
- Suitable for large-sparse matrices, cheap iterations.
- Developed convergence theory for local optimizers: As expected, fixed points of the iteration coincide with the equilibria of the ODE in general.
- Future work:
  - Further in-depth numerical experiments over benchmark problems.
  - Incorporating this algorithm into HIFOO for designing controllers.
  - Other discretization techniques of the ODE that can be efficient.

THANKS!

APPENDIX

Iteration RSVSA with Woodbury Formula

Input: ε, A, B, C, D, E_0

For $k = 1, 2, \ldots,$ let λ_{k-1} be the rightmost eigenvalue of $M(\varepsilon E_{k-1})$ with corresponding RP-compatible right and left eigenvectors x_{k-1} and y_{k-1}, and set

$$
\begin{aligned}
X_{k-1} &= (\mathrm{Re}\ x_{k-1}, \mathrm{Im}\ x_{k-1}) \in \mathbb{R}^{n\times 2} \\
Y_{k-1} &= (\mathrm{Re}\ y_{k-1}, \mathrm{Im}\ y_{k-1}) \in \mathbb{R}^{n\times 2} \\
\Xi_{k-1} &= I - \varepsilon\beta_{k-1} V_{k-1}^{\mathsf{T}} D U_{k-1} \in \mathbb{R}^{2\times 2} \\
U_k &= \left(I - \varepsilon\beta_{k-1} U_{k-1} \Xi_{k-1}^{-1} V_{k-1}^{\mathsf{T}} D\right)^{\mathsf{T}} B^{\mathsf{T}} Y_{k-1} \in \mathbb{R}^{p\times 2} \\
V_k &= \left(I + \varepsilon\beta_{k-1} D U_{k-1} \Xi_{k-1}^{-1} V_{k-1}^{\mathsf{T}}\right) C X_{k-1} \in \mathbb{R}^{m\times 2} \\
V_k &= \left(I - \varepsilon D E_{k-1}\right) C X_{k-1} \in \mathbb{R}^{m\times 2} \\
\beta_k &= 1/\|U_k V_k^{\mathsf{T}}\|_{\mathrm{F}} \in \mathbb{R} \\
E_k &= \beta_k U_k V_k^{\mathsf{T}} \in \mathbb{R}^{p\times m}
\end{aligned}
$$

Convergence of Iteration RSVSA

Definition (Fixed-point of the iteration)

A matrix $\tilde{E}$ is a fixed point of Iteration RSVSA if, supposing that $E_k = \tilde{E}$, with $M(\varepsilon\tilde{E})$ having a unique rightmost eigenvalue that is simple, then $E_{k+1} = \tilde{E}$.

Lemma (A fixed-point corresponds to an equilibrium of the ODE)

The matrix $\tilde{E}$ is a fixed point of Iteration RSVSA if and only if the ODE initialized with $E(0) = \tilde{E}$ yields $\mathrm{Re}(u(0)v(0)^*) \neq 0$ and the solution $E(t) = \tilde{E}$ for all $t \geq 0$.

Corollary (A fixed point is locally rightmost)

If $\tilde{E}$ is a fixed point of Iteration RSVSA, then there does not exist any locally differentiable path $F(t)$, with $\|F(t)\|_{\mathrm{F}} = 1$ and $F(0) = \tilde{E}$, whose rightmost eigenvalue $\kappa(t)$ has $\mathrm{Re}\,\dot{\kappa}(0) > 0$.