Computing the Real Stability Radius for Frobenius-norm Bounded Perturbations

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Motivation: Robust Stability

- Linear system $x'(t) = Ax(t)$ is stable if $x(t) \to 0$ for all starting points. Requires eigs of $A$ have negative real part.
- Measure sensitivity (robustness) of stability to perturbations as:
  - Real-life control systems are subject to noise and uncertainties in modeling.
  - Input noise should not be amplified much in applications.
  - Designing control systems by optimizing robustness measures.

Controller for Boeing 767  
International Space Station
Pseudospectra: Complex Perturbations (Reminder)

Subject of previous talk: Complex 2-norm bounded perturbations $\Delta$.

$$\sigma_\varepsilon(A) = \bigcup \{ \sigma(A + \Delta) : \|\Delta\| \leq \varepsilon \}$$

$$= \{ \lambda \in \mathbb{C} : \| (\lambda I - A)^{-1} \|_2 \geq \frac{1}{\varepsilon} \}$$

$:= \eta_A(\lambda)$ norm of the resolvent

- $\varepsilon = 10^{-1.6}$
- $\varepsilon = 10^{-1.2}$
- $\varepsilon_* = 10^{-0.8}$
- $\alpha_{\varepsilon_*}(A) = 0$
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Real/Complex Spectral Value Sets

- Spectral value sets are generalizations of pseudospectra:

  \[ \sigma_{\varepsilon}^{K,\cdot\cdot\cdot\cdot}(A, B, C, D) = \bigcup \{ \sigma(M(\Delta)) : \Delta \in K^{p \times m}, \|\Delta\| \leq \varepsilon \} . \]

  \[ M(\Delta) = A + B \Delta(I - D\Delta)^{-1}C \quad \text{for} \quad \Delta \in K^{p \times m} \]

- Different perturbation classes e.g. and norms to measure it:

  \[ \|\cdot\| = \|\cdot\|_2, \|\cdot\|_F, \ldots, \quad K = \mathbb{C} \text{ or } \mathbb{R}. \]

**Key Lemma.** Let \( \varepsilon \in \mathbb{R}^+ \) such that \( \varepsilon\|D\| < 1 \). Then

\[ \sigma_{\varepsilon}^{\mathbb{R},\cdot\cdot\cdot\cdot}(A, B, C, D) \equiv \bigcup \{ \sigma(M(\Delta)) : \Delta \in \mathbb{R}^{p \times m}, \|\Delta\| \leq \varepsilon, \text{rank}(\Delta) \leq 2 \} \]

\[ \equiv \bigcup \{ \lambda \in \mathbb{C} : \mu_{\mathbb{R},\cdot\cdot\cdot\cdot}(G(\lambda)) \geq \varepsilon^{-1} \} , \]

where \( G(\lambda) = C(\lambda I - A)^{-1}B + D \) is the transfer matrix function, \( \mu_{\mathbb{R},\cdot\cdot\cdot\cdot} \) is the real structured singular value, [Doyle 1982].
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Robust Stability Measures of Interest

- Many control systems are subject to noise that is real-valued, motivates computing robust stability measures under real-valued perturbations.
- For $\varepsilon \geq 0$, $\varepsilon \|D\| < 1$, the real structured spectral value set abscissa is
  \[
  \alpha^{\mathbb{R},\|\cdot\|}_\varepsilon (A, B, C, D) := \max \{ \Re (\lambda) : \lambda \in \sigma^{\mathbb{R},\|\cdot\|}_\varepsilon (A, B, C, D) \}
  \]
- The real stability radius for the system described by $(A, B, C, D)$ is
  \[
  \varepsilon^{\mathbb{R},\|\cdot\|}_* := \sup \left\{ \varepsilon : \varepsilon \|D\| < 1 \text{ and } \sigma^{\mathbb{R},\|\cdot\|}_\varepsilon (A, B, C, D) \subset \mathbb{C}_- \right\}.
  \]
  where $\mathbb{C}_- = \text{open left half-plane}$.
- By definition, the stability radius $\varepsilon^{\mathbb{R},\|\cdot\|}_*$ is the zero of the monotonic function $g^{\mathbb{R}} : \mathbb{R} \rightarrow \mathbb{R}$ defined by
  \[
  g^{\mathbb{R}}(\varepsilon) = \alpha^{\mathbb{R},\|\cdot\|}_\varepsilon (A, B, C, D)
  \]
Vast Literature: A Partial List

The Pseudospectra Case ($B = C = I$, $D = 0$)

- Mengi-Overton 2004: stability radius in discrete-time for $K = \mathbb{C}$
- Guglielmi-Overton 2011: a discrete iteration for $K = \mathbb{C}$
- Kressner-Vandereycken 2012: improved method for $K = \mathbb{C}$
- Guglielmi-Lubich 2013: an ODE approach for $K = \mathbb{R}$
- Guglielmi-Manetta 2014: a discrete iteration for $K = \mathbb{R}$
- Freitag-Spence 2015: implicit determinant approach for $K = \mathbb{R}$
- Kostic-Miedlar-Stolwick 15: general stability regions and $K = \mathbb{C}$

Extensions to Spectral Value Sets (general $A$, $B$, $C$, $D$)

- Gürbüzbalaban-Guglielmi-Overton 2013: a discrete iteration for complex spectral value sets and a Newton-bisection method for the complex stability radius (and $H_{\infty}$ norm), $K = \mathbb{C}$
- Benner-Voigt 2014: a closely related method, $K = \mathbb{C}$
- Mitchell-Overton 2014: a hybrid expansion contraction (HEC) method for the complex stability radius (and $H_{\infty}$ norm), $K = \mathbb{C}$
Our Contribution

- We focus on computing the real stability radius and real spectral value set abscissa.
- In 1995, an algorithm was given to extend to compute $\varepsilon_{\star}^R, \| \cdot \|_2$ to guaranteed precision [Qiu et al. 95, Sreedhar-Van Dooren-Tits 96].
- More work than the BBBS alg.; no good for large scale, does not work for F-norm.
- Our contribution: Developed Algorithm RSVSA to approximate $\varepsilon_{\star}^R, \| \cdot \|_F$, extending Algorithm SVSA (described in the previous talk).
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- More work than the BBBS alg.; no good for large scale, does not work for F-norm.
- **Our contribution**: Developed Algorithm RSVSA to approximate $\varepsilon_{\mathbb{R},\|\cdot\|_F}^{\mathbb{R},\|\cdot\|_F}$, extending Algorithm SVSA (described in the previous talk).
- **Key point**: Iterate with rank-two perturbations instead of rank-one. Inspired by [Guglielmi-Lubich 2013, Guglielmi-Manetta 2014].
Lemma

Given a smooth parametrization $\Delta(t)$ with $\|\Delta(t)\|_F = \varepsilon$ for all $t \geq 0$, suppose $\lambda(t)$ is the unique rightmost eigenvalue of $M(\Delta(t))$ and assume it is simple for all $t \geq 0$. Then $\lambda(t)$ is differentiable with

$$
\text{Re } \dot{\lambda}(t) = \frac{1}{y(t)^*x(t)} \text{Re}(u(t)^* \dot{\Delta}(t)v(t))
$$

$$
= \frac{1}{y(t)^*x(t)} \langle \dot{\Delta}(t), \text{Re}(u(t)v(t)^*) \rangle
$$

where $x(t)$ and $y(t)$ are respectively right and left RP-compatible eigenvectors associated with $\lambda(t)$, i.e., with $\|x(t)\| = \|y(t)\| = 1$ and $y(t)^*x(t)$ real and positive, and

$$
u(t) = (I - \Delta(t)D)^{-T} B^Ty(t), \quad v(t) = (I - D\Delta(t))^{-1} Cx(t).
$$
How to Make Perturbations Evolve in Time?

- How to choose the evolution of $\Delta(t)$?
  - Move to the right as quick as possible, i.e. select $E(t)$ to max $\text{Re}(\dot{\lambda}(t))$.
  - Need the magnitude right. Set $\Delta(t) = \varepsilon E(t)$ and keep $\|E(t)\|_F = 1$.

- Explicit solution to these two criteria leads to the ODE:

$$\dot{E}(t) = \text{Re}(u(t)v(t)^*) - \langle E(t), \text{Re}(u(t)v(t)^*) \rangle E(t),$$

with $u(t)$ and $v(t)$ defined by

$$u(t) = (I - \varepsilon E(t)D)^{-T} B^T y(t), \quad v(t) = (I - \varepsilon DE(t))^{-1} Cx(t)$$

where $y(t)$ and $x(t)$ are RP-compatible left and right eigenvectors, respectively, corresponding to the rightmost eigenvalue $\lambda(t)$ of $M(\varepsilon E(t))$. 
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where $y(t)$ and $x(t)$ are RP-compatible left and right eigenvectors, respectively, corresponding to the rightmost eigenvalue $\lambda(t)$ of $M(\varepsilon E(t))$. 
The Solution of the ODE

Theorem

Suppose for simplicity that for all \( t \geq 0 \), the rightmost eigenvalue \( \lambda(t) \) of \( M(E(t)) \) is unique and simple. Then, \( \Re \dot{\lambda}(t) \geq 0 \) for all \( t \) and, as \( t \to \infty \):

1. \( \lambda(t) \) converges to a limit \( \tilde{\lambda} \).
2. \( \Re (u(t)v(t)^*) \) converges to a limit \( \Re (\tilde{u}\tilde{v}^*) \).
3. \( E(t) \) converges to an “optimal” perturbation, i.e.

\[
E(t) \to \tilde{E} := \frac{\Re (\tilde{u}\tilde{v}^*)}{\| \Re (\tilde{u}\tilde{v}^*) \|_F}
\]

with \( \tilde{\lambda} \) a rightmost eigenvalue of \( M(\epsilon\tilde{E}) \) if \( \Re (\tilde{u}\tilde{v}^*) \neq 0 \).

Roughly speaking, if \( \tilde{\lambda} \) is the unique simple rightmost eigenvalue, "There is no locally differentiable path passing from \( \tilde{E} \) that could let you move to the right further with positive speed."
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with $\tilde{\lambda}$ a rightmost eigenvalue of $M(\varepsilon\tilde{E})$ if $\Re (\tilde{u}\tilde{v}^*) \neq 0$.

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The Rank of the Desired Perturbation $\tilde{E}$

By the previous theorem, the perturbation $\tilde{E} \propto \text{Re}(\tilde{u}\tilde{v}^*)$ we desire satisfies

$$
\tilde{u} = \left( I - \varepsilon \tilde{E} D \right)^{-T} B^T \tilde{y}, \quad \tilde{v} = \left( I - \varepsilon D \tilde{E} \right)^{-1} C \tilde{x}
$$

where $\tilde{y}$ and $\tilde{x}$ eigenvectors of $M(\varepsilon \tilde{E})$ corresponding to $\tilde{\lambda}$.

1. If $\tilde{\lambda} \in \mathbb{R}$, then $\text{rank} \, \text{Re}(\tilde{u}\tilde{v}^*) = 1$ as we can choose $\tilde{y}, \tilde{x}$ to be real.
2. If $\tilde{\lambda} \not\in \mathbb{R}$, set

$$
\tilde{X} = (\text{Re} \, \tilde{x}, \text{Im} \, \tilde{x}) \in \mathbb{R}^{n \times 2}, \quad \tilde{Y} = (\text{Re} \, \tilde{y}, \text{Im} \, \tilde{y}) \in \mathbb{R}^{n \times 2}
$$

so $\text{Re}(\tilde{y}\tilde{x}^*) = \tilde{Y} \tilde{X}^T$. Then

$$
\text{Re}(\tilde{u}\tilde{v}^*) = \left( I - \varepsilon \tilde{E} D \right)^{-T} B^T \tilde{Y} \tilde{X}^T C^T \left( I - \varepsilon D \tilde{E} \right)^{-T}
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with

$$
\text{rank} \, \text{Re}(\tilde{u}\tilde{v}^*) = \text{rank} (\tilde{U}\tilde{V}^T) \leq 2.
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$$= : \tilde{U}$$

$$= : \tilde{V}^T$$

with

$$\text{rank} \text{Re}(\tilde{u}\tilde{v}^*) = \text{rank} (\tilde{U}\tilde{V}^T) \leq 2.$$
Solving the ODE to Approximate $\alpha_{\varepsilon}^{\mathbb{R},\|\cdot\|_F}$

- We could integrate the ODE numerically with automatic stepsize control, but the solution $E(t)$ and its discretization will not preserve the low rank structure even if the rank is two initially and in the limit.

- Following [Guglielmi-Manetta 2014], consider the following implicit-explicit Euler discretization of the ODE with a step $h_k$:

$$E_k = E_{k-1} + h_k \left( \text{Re}(u_k v_k^*) - \langle E_k, \text{Re}(u_k v_k^*) \rangle E_{k-1} \right)$$

where $u_k$ and $v_k$ are given by

$$u_k = \left( I - \varepsilon E_{k-1} D \right)^{-T} B^T y_{k-1}, \quad v_k = \left( I - \varepsilon D E_{k-1} \right)^{-1} C x_{k-1}$$

and $y_{k-1}$ and $x_{k-1}$ are RP-compatible eigenvectors associated with the rightmost eigenvalue $\lambda_{k-1}$ of $M(\varepsilon E_{k-1})$.

- The method is consistent and converges with order 1 with respect to $h_k$. 


Solution of the Difference Equation

- Let $u_0, \nu_0$ be given vectors, let $E_0 = \text{Re} (u_0 \nu_0^*) / \| \text{Re} (u_0 \nu_0^*) \|_F$ and $h_k = 1 / \| \text{Re} (u_k \nu_k^*) \|_F$. Then the difference equation has the solution\(^1\)

$$E_k = \frac{\text{Re} (u_k \nu_k^*)}{\| \text{Re} (u_k \nu_k^*) \|_F}.$$

- So, $\text{rank}(E_k) \leq 2$ and the iteration can be done very efficiently. The main cost is computing $\lambda_k, x_k$ and $y_k$ (by 2 calls to eigs).

- There is no guarantee that $\text{Re}(\lambda_k) \geq \text{Re}(\lambda_{k-1})$, but this is generally expected to hold since the ODE discretized by the iteration has this monotonicity property. In any case, we can modify the iteration by introducing a “line search” to ensure that monotonicity holds.

---

\(^1\)as long as $\text{Re} (u_k \nu_k^*) \neq 0$ and the rightmost eigenvalue of $M(\varepsilon E_k)$ is unique and simple for all $k \geq 0$. 

Iteration RSVSA (Real Spectral Value Set Abscissa)

Input: \( \varepsilon, A, B, C, D, E_0 \)

For \( k = 1, 2, \ldots \), let \( \lambda_{k-1} \) be the rightmost eigenvalue of \( M(\varepsilon E_{k-1}) \) with corresponding RP-compatible right and left eigenvectors \( x_{k-1} \) and \( y_{k-1} \), and set\(^2\)

\[
X_{k-1} = \begin{pmatrix} \text{Re} \ x_{k-1} & \text{Im} \ x_{k-1} \end{pmatrix} \in \mathbb{R}^{n \times 2} \\
Y_{k-1} = \begin{pmatrix} \text{Re} \ y_{k-1} & \text{Im} \ y_{k-1} \end{pmatrix} \in \mathbb{R}^{n \times 2} \\
U_k = (I - \varepsilon E_{k-1} D)^T B^T Y_{k-1} \in \mathbb{R}^{p \times 2} \\
V_k = (I - \varepsilon D E_{k-1}) C X_{k-1} \in \mathbb{R}^{m \times 2} \\
\beta_k = 1/\|U_k V_k^T\|_F \in \mathbb{R} \\
E_k = \beta_k U_k V_k^T \in \mathbb{R}^{p \times m}
\]

\(^2\)well-defined if the rightmost eigenvalue of \( M(\varepsilon E_{k-1}) \) is unique and simple and \( U_k V_k^T \) is nonzero.
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For $k = 1, 2, \ldots$, let $\lambda_{k-1}$ be the rightmost eigenvalue of $M(\varepsilon E_{k-1})$ with corresponding RP-compatible right and left eigenvectors $x_{k-1}$ and $y_{k-1}$, and set\(^2\)

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X_{k-1} &= (\text{Re } x_{k-1}, \text{Im } x_{k-1}) \in \mathbb{R}^{n \times 2} \\
Y_{k-1} &= (\text{Re } y_{k-1}, \text{Im } y_{k-1}) \in \mathbb{R}^{n \times 2} \\
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\]

\(^2\)well-defined if the rightmost eigenvalue of $M(\varepsilon E_{k-1})$ is unique and simple and $U_k V_k^T$ is nonzero.
Iterations of Algorithm RSVSA: Example 1

- black: $\sigma(A)$
- green: boundary of $\sigma_\epsilon(A, B, C, D)$
- blue: iterates of Algorithm SVSA (complex perturbations)
- gray: $\sigma_\epsilon^{\mathbb{R},\|\cdot\|_F}(A, B, C, D)$, sampled from 6000 random perturbations
- red: iterates of Algorithm RSVSA (real perturbations, Frob. norm)
Iterations of Algorithm RSVSA: Example 2

black: $\sigma(A)$
green: boundary of $\sigma_\epsilon(A, B, C, D)$
blue: iterates of Algorithm SVSA (complex perturbations)
gray: $\sigma^{\mathbb{R},\|\cdot\|_F}(A, B, C, D)$, sampled from 6000 random perturbations
red: iterates of Algorithm RSVSA (real perturbations, Frob. norm)
Computing the real stability radius with HEC

- By definition, the real stability radius $\varepsilon^*$ is equal to the $\varepsilon$ level which results in $\alpha_{\varepsilon}^{\mathbb{R},\|\cdot\|_F}(A, B, C, D) = 0$.

- Now that we have an efficient alg. to compute $\alpha_{\varepsilon}^{\mathbb{R},\|\cdot\|_F}(A, B, C, D)$ for a given $\varepsilon$, we can use the Hybrid Expansion-Contraction (HEC) approach from [Mitchell-Overton 2014] to compute $\varepsilon^*$ which extends with minor modifications.

- Recap from previous talk: HEC improves the bisection-Newton approach with some desirable properties:
  - Adaptively damped/accelerated Newton scheme, provably quadratically convergent that is visible in the experiments.
  - Often/usually finds the true value of the H-infinity norm.
  - When it does not, usually finds good approximations.
  - Also very fast in practice.
**Step 0:** Generate a point in the right half-plane and contract it back to the imaginary axis, gives an upper bound $\varepsilon_1 \approx 0.71$ of the real stability radius $\varepsilon_\star$.

**Step 1:** Expand to compute abscissa, $\alpha_{\varepsilon_1} \approx 56.06$. 
**Step 2**: Contracting previous iterate to imaginary axis, gives a new upper bound $\varepsilon_2 \approx 0.42$, iterates shown for computing $\alpha\varepsilon_2 \approx 0.43$. 
Step 3:  Left: Contracting previous iterate gives a new upper bound \( \varepsilon_3 \approx 0.40 \), iterates shown for computing \( \alpha_{\varepsilon_3} \approx 0.0013 \).  Right: Zoom in.
Summary and future work

- Presented a new algorithm for computing the real spectral value set abscissa and real stability radius under Frobenius-norm bounded perturbations.
- Suitable for large-sparse matrices, cheap iterations.
- Developed convergence theory for local optimizers: As expected, fixed points of the iteration coincide with the equilibria of the ODE in general.
- Future work:
  - Further in-depth numerical experiments over benchmark problems.
  - Incorporating this algorithm into HIFOO for designing controllers.
  - Other discretization techniques of the ODE that can be efficient.

THANKS!
Iteration RSVSA with Woodbury Formula

Input: $\epsilon, A, B, C, D, E_0$

For $k = 1, 2, \ldots$, let $\lambda_{k-1}$ be the rightmost eigenvalue of $M(\epsilon E_{k-1})$ with corresponding RP-compatible right and left eigenvectors $x_{k-1}$ and $y_{k-1}$, and set

\[
X_{k-1} = (\text{Re } x_{k-1}, \text{Im } x_{k-1}) \in \mathbb{R}^{n \times 2}
\]

\[
Y_{k-1} = (\text{Re } y_{k-1}, \text{Im } y_{k-1}) \in \mathbb{R}^{n \times 2}
\]

\[
\Xi_{k-1} = I - \epsilon \beta_{k-1} V_{k-1}^T D U_{k-1} \in \mathbb{R}^{2 \times 2}
\]

\[
U_k = \left( I - \epsilon \beta_{k-1} U_{k-1} \Xi_{k-1}^{-1} V_{k-1}^T D \right)^T B^T Y_{k-1} \in \mathbb{R}^{p \times 2}
\]

\[
V_k = \left( I + \epsilon \beta_{k-1} D U_{k-1} \Xi_{k-1}^{-1} V_{k-1}^T \right) C X_{k-1} \in \mathbb{R}^{m \times 2}
\]

\[
\beta_k = \frac{1}{\| U_k V_k^T \|_F} \in \mathbb{R}
\]

\[
E_k = \beta_k U_k V_k^T \in \mathbb{R}^{p \times m}
\]
Convergence of Iteration RSVSA

Definition (Fixed-point of the iteration)

A matrix $\tilde{E}$ is a fixed point of Iteration RSVSA if, supposing that $E_k = \tilde{E}$, with $M(\varepsilon\tilde{E})$ having a unique rightmost eigenvalue that is simple, then $E_{k+1} = \tilde{E}$.

Lemma (A fixed-point corresponds to an equilibrium of the ODE)

The matrix $\tilde{E}$ is a fixed point of Iteration RSVSA if and only if the ODE initialized with $E(0) = \tilde{E}$ yields $\Re(u(0)v(0)^*) \neq 0$ and the solution $E(t) = \tilde{E}$ for all $t \geq 0$.

Corollary (A fixed point is locally rightmost)

If $\tilde{E}$ is a fixed point of Iteration RSVSA, then there does not exist any locally differentiable path $F(t)$, with $\|F(t)\|_F = 1$ and $F(0) = \tilde{E}$, whose rightmost eigenvalue $\kappa(t)$ has $\Re\kappa(0) > 0$. 